Quantum mechanics of photons and Maxwell's equations

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Motto

“The main principle of the present work is the idea that, since matter and light both possess the dual characters of particle and wave, a similar mathematical treatment should be applied to both, and that this has not been yet done as fully as should be possible”.

Charles G. Darwin 1932

The Schrödinger equation for photons

Weyl equation for (massless) neutrinos (spin 1/2)

$$i\hbar\partial_t\psi = c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\psi \quad \text{or} \quad i\partial_t\psi = -ic\,(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi$$

Analogous equation for photons (spin 1)

$$i\partial_t\boldsymbol{F} = -ic\,(\boldsymbol{S}\cdot\boldsymbol{\nabla})\,\boldsymbol{F}$$

$$\boldsymbol{S}\cdot\boldsymbol{\nabla} = \begin{bmatrix} 0 & -i\partial_z & i\partial_y \\ i\partial_z & 0 & -i\partial_x \\ -i\partial_y & i\partial_x & 0 \end{bmatrix} = i\boldsymbol{\nabla}\times$$

Splitting into real and imaginary parts

$$i\partial_t \boldsymbol{F} = c\boldsymbol{\nabla} \times \boldsymbol{F}$$

$$\boldsymbol{F} = \Re(\boldsymbol{F}) + i\Im(\boldsymbol{F})$$

$$\partial_t \Re(\boldsymbol{F}) = c\boldsymbol{\nabla} \times \Im(\boldsymbol{F}) \qquad \partial_t \Im(\boldsymbol{F}) = -c\boldsymbol{\nabla} \times \Re(\boldsymbol{F})$$

$$\Re(\boldsymbol{F}) = \sqrt{\epsilon}\boldsymbol{E} = \frac{\boldsymbol{D}}{\sqrt{\epsilon}} \qquad \Im(\boldsymbol{F}) = \sqrt{\mu}\boldsymbol{H} = \frac{\boldsymbol{B}}{\sqrt{\mu}}$$

$$\partial_t \boldsymbol{D} = \boldsymbol{\nabla} \times \boldsymbol{H} \qquad \partial_t \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

Relativistic quantum mechanics of photons

General solution of the Schrödinger equation for photons

$$\boldsymbol{F}(\boldsymbol{r},t)=\sqrt{\hbar c}\int\frac{d^3k}{(2\pi)^{3/2}}\boldsymbol{e}(\boldsymbol{k})\left[f_L(\boldsymbol{k})e^{-i\omega_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{r}}+f_R^*(\boldsymbol{k})e^{i\omega_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

Polarization vector obeys the (Maxwell) equation

$$-i\omega\boldsymbol{e}(\boldsymbol{k})=c\boldsymbol{k}\times\boldsymbol{e}(\boldsymbol{k})\qquad\boldsymbol{e}^*\cdot\boldsymbol{e}=1$$

The amplitudes $f_L(\boldsymbol{k})$ and $f_R(\boldsymbol{k})$ are the wave functions in momentum space

Quantum operators in $\boldsymbol{k}$-space

Generators of Poincaré transformations

The transformations $F_i'(\boldsymbol{r}', t') = O_i^j F_j(\boldsymbol{r}, t)$ must preserve the form of the photon wave equation

Time translation: Energy$=\hbar\omega$

Space translation: Momentum$=\hbar\boldsymbol{k}$

Rotation: Angular momentum$=i\hbar\boldsymbol{k}\times\mathcal{D}_{\boldsymbol{k}} + \hat{\chi}\hbar\boldsymbol{k}/k$

Lorentz transformation: Boost$=i\hbar\omega\mathcal{D}_{\boldsymbol{k}}$

Helicity operator $\hat{\chi}$ takes on two values ± 1

$$\mathcal{D}_{\boldsymbol{k}} = \boldsymbol{\nabla}_k - i\hat{\chi}\boldsymbol{\alpha}(\boldsymbol{k}) \qquad \boldsymbol{\nabla}_k \times \boldsymbol{\alpha}(\boldsymbol{k}) = -\boldsymbol{k}/k^3$$

Quantum-classical correspondence

QM average values agree with classical expressions

Energy$=\langle\hbar\omega\rangle = \int d^3r\,[\boldsymbol{D}^2/2\epsilon + \boldsymbol{B}^2/2\mu]$

Momentum$=\langle\hbar\boldsymbol{k}\rangle = \int d^3r\,[\boldsymbol{D}\times\boldsymbol{B}]$

Angular momentum$=\langle i\hbar\boldsymbol{k}\times\mathcal{D}_{\boldsymbol{k}} + \hat{\chi}\hbar\boldsymbol{k}/k\rangle$
$= \int d^3r\,[\boldsymbol{r}\times(\boldsymbol{D}\times\boldsymbol{B})]$

Lorentz transformation: Boost$=\langle i\hbar\omega\mathcal{D}_{\boldsymbol{k}}\rangle$
$= \int d^3r\,\boldsymbol{r}\,[\boldsymbol{D}^2/2\epsilon + \boldsymbol{B}^2/2\mu]$

Second quantization

Quantized electromagnetic field operator

$$\hat{\boldsymbol{F}}(\boldsymbol{r},t) = \sqrt{\hbar c}\int \frac{d^3k}{(2\pi)^{3/2}}\boldsymbol{e}(\boldsymbol{k})\left[a_L(\boldsymbol{k})e^{-i\omega_{\boldsymbol{k}}t+i\boldsymbol{k}\cdot\boldsymbol{r}} + a_R^\dagger(\boldsymbol{k})e^{i\omega_{\boldsymbol{k}}t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

Photons do not have a conserved quantum number (charge, lepton number, etc.)

Formally, right-handed and left-handed photons are in the particle-antiparticle relation but we can make all their superpositions that create photon states with arbitrary polarization

$$|\Psi_{\text{one photon}}\rangle = \int \frac{d^3k}{k}\left[f_L(\boldsymbol{k})a_L^\dagger(\boldsymbol{k}) + f_R(\boldsymbol{k})a_R^\dagger(\boldsymbol{k})\right]|0\rangle$$

How come classical EM field?

How come Maxwell's equations?

Key property: Number of photons N

$$N = \frac{\textbf{Power} \times \textbf{Time}}{\textbf{Photon energy}} = 7.5 \times 10^{31} \frac{\textbf{P[in Watt]} \times \textbf{T[in Sec]}}{\nu\textbf{[in Hertz]}}$$

Small WiFi router (50mW) operating at 2.4GHz sends 3×10^{22} photons per second

Coherent states

Note that the average field in any state with fixed number of photons vanishes $\langle\Psi_N|\hat{F}|\Psi_N\rangle = 0$

It is obvious that one cannot precisely control N at the level of 10^{22}

Randomly produced photons are characterized by the Poisson distribution $\langle N\rangle^k/k!e^{-\langle N\rangle}$

The Poissonian quantum-mechanical state is:

$$|\alpha\rangle = \sum_1^\infty \frac{\alpha^k}{\sqrt{k!}}|k\rangle \qquad |\alpha|^2 = \langle N\rangle$$

This state is called coherent state

Average field

Assume that a device (say a router) produces photons characterized by the creation operator

$$a_f^\dagger = \int \frac{d^3k}{k} \left[f_L(\boldsymbol{k}) a_L^\dagger(\boldsymbol{k}) + f_R(\boldsymbol{k}) a_R^\dagger(\boldsymbol{k}) \right]$$

The classical electromagnetic field is the average value obtained from the complex average value $\langle \hat{\boldsymbol{F}} \rangle_f$ of $\hat{\boldsymbol{F}}$ calculated in the coherent state corresponding to $a_f^\dagger$

$$\langle \hat{\boldsymbol{F}} \rangle_f = \sqrt{\langle N \rangle \hbar c} \int \frac{d^3k}{(2\pi)^{3/2}} \boldsymbol{e}(\boldsymbol{k}) \left[f_L(\boldsymbol{k}) e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{r}} + f_R^*(\boldsymbol{k}) e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right]$$

Closing the argument

In classical electrodynamics classical sources produce classical electromagnetic field

What state $|\boldsymbol{\Psi}\rangle$ of the quantum electromagnetic field is produced by a classical current $\boldsymbol{J}^{\mu}(\boldsymbol{r},t)$?

The answer is obtained from the formula

$$|\boldsymbol{\Psi}\rangle = \boldsymbol{T}\exp\left(-i\int d^4x\,\hat{\boldsymbol{A}}_{\mu}(\boldsymbol{r},t)\boldsymbol{J}^{\mu}(\boldsymbol{r},t)\right)|0\rangle$$

The state $|\boldsymbol{\Psi}\rangle$ is a coherent state and the average field in this state is the same as the one obtained from the classical theory!

Summary

Maxwell's equations can be derived from quantum mechanics of photons in the classical limit

The classical limit means here not $\hbar \to 0$ but

a very large average number of photons

obeying the Poisson distribution

Classical fields are identified as

expectation values of the quantum field operators